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GEOMETRIC PROPERTIES OF SOLUTIONS OF
A CLASS OF DIFFERENTIAL EQUATIONS
(Study on Applications for Fractional
Calculus Operators in Univalent Function
Theory)

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GEOMETRIC PROPERTIES OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

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Abstract

The main object of this paper is to investigate several geometric properties of the solutions of the second-order linear differential equation:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where the functions $a(z)$ and $b(z)$ are analytic in the open unit disk U . Relevant connections of the results presented in this paper with those given earlier by (for example) M.S. Robertson, S.S. Miller, and H. Saitoh are also considered.

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1. Introduction

Let \mathcal{A} denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let \mathcal{S} , \mathcal{S}^* , and $\mathcal{S}^*(\alpha)$ denote the subclasses of \mathcal{A} consisting of functions which are, respectively, *univalent*, *starlike with respect to the origin*, and *starlike of order α* in \mathbb{U} ($0 \leq \alpha < 1$). Thus, by definition, we have (see, for details, [2] and [8]; see also [7] and [11])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\} \quad (1.2)$$

and

$$\mathcal{S}^* := \mathcal{S}^*(\alpha)|_{\alpha=0} = \mathcal{S}^*(0). \quad (1.3)$$

For functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ ($z \in \mathbb{U}$), we define the Schwarzian derivative of $f(z)$ by

$$S(f, z) := \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \quad (1.4)$$

$$(f \in \mathcal{A}; f'(z) \neq 0 \text{ } (z \in \mathbb{U})).$$

We begin by recalling the following result of Miller [4].

Theorem A (Miller [4]). *Let the function $p(z)$ be analytic in \mathbb{U} with*

$$|zp(z)| < 1 \quad (z \in \mathbb{U}).$$

Also let $v(z)$ denote the unique solution of the following initial-value problem:

$$v''(z) + p(z)v(z) = 0 \quad (v(0) = 0; v'(0) = 1) \quad (1.5)$$

in \mathbb{U} . Then

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}) \quad (1.6)$$

and $v(z)$ is a starlike conformal map of the unit disk \mathbb{U} .

Theorem A is related rather closely to some earlier results of Robertson [9] and Nehari [6], which we recall here as Theorem B and Theorem C below.

Theorem B (Robertson [9]). *Let $zp(z)$ be analytic in \mathbb{U} and*

$$\Re \{ z^2 p(z) \} \leq \frac{\pi^2}{4} |z|^2 \quad (z \in \mathbb{U}). \quad (1.7)$$

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Then the unique solution $W = W(z)$ of the following initial-value problem:

$$W''(z) + p(z)W(z) = 0 \quad (W(0) = 0; W'(0) = 1) \quad (1.8)$$

is univalent and starlike in \mathbb{U} . The constant $\frac{\pi^2}{4}$ in the inequality (1.7) is the best possible.

Theorem C (Nehari [6]). If $f \in \mathcal{A}$ satisfies the following inequality involving its Schwarzian derivative defined by (1.4):

$$|S(f, z)| \leq \frac{\pi^2}{2} \quad (z \in \mathbb{U}), \quad (1.9)$$

then $f \in \mathcal{S}$. The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{e^{i\pi z} - 1}{i\pi} \quad (i := \sqrt{-1}). \quad (1.10)$$

Remark 1. By setting

$$p(z) = \frac{1}{2}S(f, z) \quad (z \in \mathbb{U}) \quad (1.11)$$

and using (1.9), we obtain the inequality (1.7). Obviously, therefore, the hypothesis in Theorem C is stronger than that in Theorem B.

In the present paper, we aim at investigating several *geometric* properties of the solutions of the following initial-value problem which involves a general family of second-order linear differential equations:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \quad (1.12)$$

$$(w(0) = 0; w'(0) = 1),$$

where the functions $a(z)$ and $b(z)$ are analytic in \mathbb{U} (see [3]). We also show how our results are related to those of (for example) Robertson [9], Miller [4], and Saitoh [10].

2. A Class of Bounded Functions

Let \mathcal{B}_J denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (2.1)$$

analytic in \mathbb{U} , for which

$$|w(z)| < J \quad (z \in \mathbb{U}; J > 0). \quad (2.2)$$

If $g(z) \in \mathcal{B}_J$, then we can show (by using the Schwarz lemma [1]) that the function $w(z)$ defined by

$$w(z) := z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt \quad (2.3)$$

is also in the class \mathcal{B}_J . Thus, in terms of derivatives, we have

$$\left| \frac{1}{2}w(z) + zw'(z) \right| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}). \quad (2.4)$$

Furthermore, by letting

$$h(u, v) := \frac{1}{2}u + v, \quad (2.5)$$

we can rewrite (2.4) in the form:

$$|h(w(z), zw'(z))| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}). \quad (2.6)$$

In this section, we show that the implication (2.6) holds true for functions $h(u, v)$ in the class \mathcal{H}_J given by Definition 1 below (see also [5]).

Definition 1. Let \mathcal{H}_J be the class of complex functions $h(u, v)$ satisfying each of the following conditions:

- (i) $h(u, v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| < J$ ($J > 0$);
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ whenever

$$(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D} \quad (\theta \in \mathbb{R}; K \geq J > 0).$$

Example 1. It is easily seen that the function

$$h(u, v) = \alpha u + v \quad (\Re(\alpha) \geq 0; \mathbb{D} = \mathbb{C} \times \mathbb{C}) \quad (2.7)$$

is in the class \mathcal{H}_J .

Definition 2. Let $h \in \mathcal{H}_J$ with the corresponding domain \mathbb{D} . We denote by $\mathcal{B}_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in \mathbb{U} and satisfy each of the following conditions:

- (i) $(w(z), zw'(z)) \in \mathbb{D}$;
- (ii) $|h(w(z), zw'(z))| < J$ ($z \in \mathbb{U}; J > 0$).

The function class $\mathcal{B}_J(h)$ is not empty. Indeed, for any given function $h \in \mathcal{H}_J$, we have

$$w(z) = c_1 z \in \mathcal{B}_J(h) \quad (2.8)$$

for sufficiently small $|c_1|$ depending on h .

Theorem D (Saitoh [10]). For any $h \in \mathcal{H}_J$,

$$\mathcal{B}_J(h) \subset \mathcal{B}_J \quad (h \in \mathcal{H}_J; J > 0).$$

Remark 2. Theorem D shows that, if $h \in \mathcal{H}_J$ (with the corresponding domain \mathbb{D}) and if $w(z)$, given by (2.1), is analytic in \mathbb{U} and

$$(w(z), zw'(z)) \in \mathbb{D},$$

then the implication (2.4) holds true.

Theorem D leads us immediately to the following result, which was also given by Saitoh [10].

Theorem E (Saitoh [10]). *Let $h \in \mathcal{H}_J$ and let the function $b(z)$ be analytic in \mathbb{U} with*

$$|b(z)| < J \quad (z \in \mathbb{U}; J > 0).$$

If the initial-value problem:

$$h(w(z), zw'(z)) = b(z) \quad (w(0) = 0) \quad (2.9)$$

has a solution $w(z)$ analytic in \mathbb{U} , then

$$|w(z)| < J \quad (z \in \mathbb{U}; J > 0).$$

3. Main Results and Their Consequences

One of our *main* results is contained in the following theorem.

Theorem 1. *Let the functions $a(z)$ and $b(z)$ be analytic in \mathbb{U} with*

$$\left| z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} \right| < J \quad (z \in \mathbb{U}; J > 0) \quad (3.1)$$

and

$$\Re\{za(z)\} > -2J \quad (z \in \mathbb{U}; J > 0). \quad (3.2)$$

Also let $w(z)$ denote the solution of the initial-value problem (1.12) in \mathbb{U} . Then

$$1 - J - \frac{1}{2}\Re\{za(z)\} < \Re\left(\frac{zw'(z)}{w(z)}\right) < 1 + J - \frac{1}{2}\Re\{za(z)\} \quad (3.3)$$

$$(z \in \mathbb{U}; J > 0).$$

Proof. First of all, by means of the transformation:

$$w(z) = \exp\left(-\frac{1}{2}\int_0^z a(t) dt\right) \cdot v(z), \quad (3.4)$$

we can rewrite the initial-value problem (1.12) in the *normal* form:

$$v''(z) + \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} v(z) = 0 \quad (3.5)$$

$$(v(0) = 0; v'(0) = 1).$$

If we now put

$$u(z) = \frac{zv'(z)}{v(z)} - 1 \quad (z \in \mathbb{U}), \quad (3.6)$$

then $u(z)$ is analytic in \mathbb{U} with $u(0) = 0$, and (3.5) becomes

$$[u(z)]^2 + u(z) + zu'(z) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} \quad (u(0) = 0) \quad (3.7)$$

or, equivalently,

$$h(u(z), zu'(z)) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} \quad (u(0) = 0), \quad (3.8)$$

where, for convenience,

$$h(\xi, \eta) := \xi^2 + \xi + \eta. \quad (3.9)$$

It is easily observed from (3.1), (3.2), and (3.8) that $h(\xi, \eta) \in \mathcal{H}_J$, that is, that

- (i) $h(\xi, \eta)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| = 0 < J$ ($J > 0$);
- (iii) For $(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D}$ ($\theta \in \mathbb{R}$; $K \geq J > 0$),

$$\begin{aligned} |h(Je^{i\theta}, Ke^{i\theta})| &= |J^2e^{2i\theta} + Je^{i\theta} + Ke^{i\theta}| \\ &= |J^2e^{i\theta} + J + K| \geq J. \end{aligned}$$

Thus, by applying Theorem E, we find from the hypothesis (3.1) of Theorem 1 that

$$|u(z)| < J \quad (z \in \mathbb{U}; J > 0),$$

which, in view of the relationship (3.6), yields

$$1 - J < \Re \left(\frac{zv'(z)}{v(z)} \right) < 1 + J \quad (z \in \mathbb{U}; J > 0). \quad (3.10)$$

Next, by logarithmically differentiating (3.4) in its *equivalent* form:

$$v(z) = \exp \left(\frac{1}{2} \int_0^z a(t) dt \right) \cdot w(z),$$

we have

$$\frac{zv'(z)}{v(z)} = \frac{zw'(z)}{w(z)} + \frac{1}{2}za(z), \quad (3.11)$$

so that (3.10) becomes

$$\begin{aligned} 1 - J < \Re \left(\frac{zw'(z)}{w(z)} \right) + \frac{1}{2}\Re\{za(z)\} < 1 + J \\ (z \in \mathbb{U}; J > 0), \end{aligned} \quad (3.12)$$

which obviously yields the assertion (3.3) of Theorem 1.

Remark 3. If, in Theorem 1, we have

$$\Re\{za(z)\} \leq 2(1-J) \quad (z \in \mathbb{U}; J > 0), \quad (3.13)$$

so that

$$0 \leq 1 - J - \frac{1}{2}\Re\{za(z)\} < 1 \quad (z \in \mathbb{U}; J > 0), \quad (3.14)$$

then the assertion (3.3) immediately yields

$$w(z) \in \mathcal{S}^* \left(1 - J - \frac{1}{2}\Re\{za(z)\} \right)$$

in conjunction with the definition (1.2).

Example 2. If we let

$$a(z) = -2Jz \quad \text{and} \quad b(z) = J^2 z^2 \quad (J > 0) \quad (3.15)$$

in Theorem 1, then the solution of the initial-value problem:

$$\begin{aligned} w''(z) - 2Jzw'(z) + J^2 z^2 w(z) &= 0 \\ (w(0) = 0; w'(0) &= 1) \end{aligned} \quad (3.16)$$

is given by

$$w(z) = \frac{1}{\sqrt{J}} \exp\left(\frac{1}{2}Jz^2\right) \cdot \sin(z\sqrt{J}). \quad (3.17)$$

In this case, if we further assume that

$$0 < J \leq \frac{1}{2},$$

then

$$w(z) \in \mathcal{S}^*(1-2J) \quad \left(0 < J \leq \frac{1}{2}\right),$$

so that, in particular, we have

$$\begin{aligned} J = \frac{1}{2} : w(z) &= \sqrt{2} \exp\left(\frac{1}{4}z^2\right) \cdot \sin\left(\frac{z}{\sqrt{2}}\right) \in \mathcal{S}^*, \\ J = \frac{1}{3} : w(z) &= \sqrt{3} \exp\left(\frac{1}{6}z^2\right) \cdot \sin\left(\frac{z}{\sqrt{3}}\right) \in \mathcal{S}^*\left(\frac{1}{3}\right), \\ J = \frac{1}{4} : w(z) &= 2 \exp\left(\frac{1}{8}z^2\right) \cdot \sin\left(\frac{z}{2}\right) \in \mathcal{S}^*\left(\frac{1}{2}\right), \end{aligned}$$

and so on.

Example 3. For

$$a(z) = -2Jz \quad \text{and} \quad b(z) = \lambda \quad (J > 0; \lambda \in \mathbb{C}), \quad (3.18)$$

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the initial-value problem (1.12) becomes

$$w''(z) - 2Jzw'(z) + \lambda w(z) = 0 \quad (3.19)$$

$$(w(0) = 0; w'(0) = 1),$$

which, under the transformation:

$$w(z) = \exp\left(\frac{1}{2}Jz^2\right) \cdot v(z), \quad (3.20)$$

assumes the *normal* form:

$$v''(z) + (\lambda + J - J^2z^2)v(z) = 0 \quad (3.21)$$

$$(v(0) = 0; v'(0) = 1).$$

Remark 4. In their special case when $J = \frac{1}{2}$, the differential equations in (3.19) and (3.21) can be identified with such classical differential equations as Hermite's equation and Weber's equation, respectively (*cf.*, *e.g.*, [1] and [12]).

Next we prove the following result for the solution of the initial-value problem (3.21).

Theorem 2. *If*

$$|\lambda + J - J^2z^2| < T \quad (z \in \mathbb{U}; J, T > 0), \quad (3.22)$$

then the solution $v(z)$ of the initial-value problem (3.21) satisfies the inequality:

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < T \quad (z \in \mathbb{U}; T > 0). \quad (3.23)$$

Proof. Just as in our demonstration of Theorem 1, the function $u(z)$, given by (3.6), is analytic in \mathbb{U} , $u(0) = 0$, and [*cf.* Equation (3.7)]

$$h(u(z), zu'(z)) = -z^2(\lambda + J - J^2z^2) \quad (u(0) = 0), \quad (3.24)$$

where $h(\xi, \eta)$ is defined, as before, by (3.9).

Now it is easily seen from (3.22) and (3.24) that $h(\xi, \eta) \in \mathcal{H}_T$, that is, that

- (i) $h(\xi, \eta)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| = 0 < T$ ($T > 0$);
- (iii) For $(Te^{i\theta}, Ke^{i\theta}) \in \mathbb{D}$ ($\theta \in \mathbb{R}$; $K \geq T > 0$),

$$|h(Te^{i\theta}, Ke^{i\theta})| \geq T.$$

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By applying Theorem E, we thus find from the hypothesis (3.22) of Theorem 2 that

$$|u(z)| < T \quad (z \in \mathbb{U}; T > 0),$$

which, in view of the relationship (3.6) again, leads us at once to the assertion (3.23) of Theorem 2.

Remark 5. If $0 < T \leq 1$, then Theorem 2 yields the following geometric property:

$$v(z) \in \mathcal{S}^*(1-T) \quad (0 < T \leq 1)$$

for the solution $v(z)$ of the initial-value problem (3.21).

By putting $J = \frac{1}{2}$ and $T = 1$ in Theorem 2, we obtain the following known result.

Corollary (Saitoh [10]). *If*

$$\left| \lambda + \frac{1}{2} - \frac{1}{4}z^2 \right| < 1 \quad (z \in \mathbb{U}), \quad (3.25)$$

then the solution $v(z)$ of Weber's differential equation:

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{1}{4}z^2 \right) v(z) = 0 \quad (3.26)$$

$$(v(0) = 0; v'(0) = 1)$$

is starlike in \mathbb{U} .

Remark 6. The solutions of Weber's differential equation in (3.26) are expressed as the parabolic cylinder (or Weber's) function $D_\lambda(z)$ defined by (cf., e.g., [1, p. 39 et seq.]

$$D_\lambda(z) := 2^{\lambda/2} \sqrt{\pi} \exp\left(-\frac{1}{4}z^2\right) \left[\frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)} {}_1F_1\left(-\frac{\lambda}{2}; \frac{1}{2}; \frac{1}{2}z^2\right) - \frac{z\sqrt{2}}{\Gamma\left(\frac{1-\lambda}{2}\right)} {}_1F_1\left(\frac{1-\lambda}{2}; \frac{3}{2}; \frac{1}{2}z^2\right) \right], \quad (3.27)$$

where ${}_1F_1(\alpha; \gamma; z)$ denotes the confluent hypergeometric function (see, for details, [1] and [12]).

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